

# Weighted Sobolev spaces of radially symmetric functions

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## Abstract

We prove dilation invariant inequalities involving radial functions, poliharmonic operators and weights that are powers of the distance from the origin. Then we discuss the existence of extremals and in some cases we compute the best constants.

**Keywords:** Rellich inequality, Sobolev inequality, Caffarelli-Kohn-Nirenberg inequality, weighted biharmonic operator, dilation invariance.

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## 1 Introduction

The starting point of the present paper is the inequality

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx \geq c_{\alpha,k,p} \int_{\mathbb{R}^n} |x|^{\alpha-kp} |u|^p dx \quad \forall u \in C_{c,r}^k(\mathbb{R}^n \setminus \{0\}). \quad (1.1)$$

Here  $n \geq 2$  and  $k \geq 1$  are integers,  $\alpha \in \mathbb{R}$ ,  $p > 1$ ,  $C_{c,r}^k(\mathbb{R}^n \setminus \{0\})$  is the space of radially symmetric functions in  $C_c^k(\mathbb{R}^n \setminus \{0\})$ , and

$$\nabla^k = \begin{cases} \Delta^m & \text{if } k = 2m \text{ is even,} \\ \nabla \Delta^m & \text{if } k = 2m + 1 \text{ is odd.} \end{cases}$$

Let us briefly describe our main results and our motivations.

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First we find out the class of parameters  $\alpha, k$  and  $p$  such that (1.1) holds with a positive best constant  $c_{\alpha,k,p}$ . When  $c_{\alpha,k,p} > 0$  we define the space  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  as the completion of  $C_{c,r}^k(\mathbb{R}^n \setminus \{0\})$  with respect to the norm

$$\|u\|_{k,\alpha} = \left( \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx \right)^{1/p}. \quad (1.2)$$

Then clearly  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \hookrightarrow L^p(\mathbb{R}^n; |x|^{\alpha-kp} dx)$ . Notice that  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  is the natural ambient space in dealing with radially symmetric solutions to poliharmonic problems with weights. Having this application in mind, we study dilation invariant inequalities of the type

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx \geq S_{k,q,j}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-\beta_{k-j,q}} |\nabla^j u|^q dx \right)^{p/q} \quad (1.3)$$

for  $u \in \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ , and we show that the best constant  $S_{k,q,j}(\alpha)$  is positive and achieved. Here  $j = 0, \dots, k-1$  is an integer,  $\nabla^0 u = u$ ,  $q > p$  is given, and

$$\beta_{k-j,q} = n - q \frac{n - (k-j)p + \alpha}{p}. \quad (1.4)$$

We remark that (1.1) is closely related to the double weighted Hardy–Littlewood–Sobolev inequality (see e.g. Stein and Weiss [22] and Lieb [13]). The standard Hardy inequality is recovered by choosing  $k = 1$ : it is well known that

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla u|^p dx \geq \left| \frac{n + \alpha}{p} - 1 \right|^p \int_{\mathbb{R}^n} |x|^{\alpha-p} |u|^p dx \quad \forall u \in C_c^k(\mathbb{R}^n \setminus \{0\}) \quad (1.5)$$

holds with a sharp constant in the right hand side.

Second order dilation-invariant inequalities have been largely studied since 1954, when Rellich showed that

$$\int_{\mathbb{R}^n} |\Delta u|^2 dx \geq \left( \frac{n(n-4)}{4} \right)^2 \int_{\mathbb{R}^n} |x|^{-4} |u|^2 dx \quad \forall u \in C_{c,r}^2(\mathbb{R}^n \setminus \{0\}).$$

In the Hilbertian case  $p = 2$ , Ghoussoub and Moradifam proved in [12] that  $c_{\alpha,2,2} = |(n-4+\alpha)(n-\alpha)/4|^2$  (see also [6], where a different approach is used).

Only partial results are available if  $p \neq 2$  or  $k \geq 3$ , see for instance Mitidieri [15] (in a non-radial setting) and Adimurthi and Santra [2]. Related inequalities can be found in the above quoted papers, in [1], [7], [10], [16] and in the references therein.

In the present paper we compute  $c_{\alpha,k,p}$  for any  $\alpha, k$  and  $p$ . We put

$$H_\alpha = \frac{n+\alpha}{p} - 1, \quad \gamma_{\alpha,h} = \left( \frac{n+\alpha}{p} - h \right) \left( n - 2 + h - \frac{n+\alpha}{p} \right), \quad (1.6)$$

where  $h = 2, \dots, k$  is an integer. Notice that  $|H_\alpha|^p$  is the Hardy constant.

**Theorem 1.1** *Let  $\alpha \in \mathbb{R}$ ,  $p > 1$  and  $k \geq 2$ . The best constant in (1.1) is given by*

$$c_{\alpha,k,p} = \begin{cases} \prod_{h=1}^m |\gamma_{\alpha,2h}|^p & \text{if } k = 2m \text{ is even,} \\ |H_\alpha|^p \prod_{h=1}^m |\gamma_{\alpha,2h+1}|^p & \text{if } k = 2m + 1 \text{ is odd.} \end{cases} \quad (1.7)$$

In case  $k = 2$  Theorem 1.1 implies that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx \geq \left| \left( \frac{n+\alpha}{p} - 2 \right) \left( n - \frac{n+\alpha}{p} \right) \right|^p \int_{\mathbb{R}^n} |x|^{\alpha-2p} |u|^p dx \quad (1.8)$$

for any  $u \in C_{c,r}^k(\mathbb{R}^n \setminus \{0\})$ . In fact (1.8) is a corollary of the Hardy inequality and of the next result. Here we agree that  $\Delta^0 u = u$ .

**Theorem 1.2** *Let  $\alpha \in \mathbb{R}$ ,  $p > 1$  and let  $m \geq 1$  be a given integer. Then*

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta^m u|^p dx \geq \left| n - \frac{n+\alpha}{p} \right|^p \int_{\mathbb{R}^n} |x|^{\alpha-p} |\nabla(\Delta^{m-1} u)|^p dx \quad (1.9)$$

for any  $u \in C_{c,r}^{2m}(\mathbb{R}^n \setminus \{0\})$ . The constant in the right hand side is sharp.

Notice that in the singular case  $\alpha = 2p - n$  the best constant in (1.8) vanishes, while

$$\int_{\mathbb{R}^n} |x|^{2p-n} |\Delta^m u|^p dx \geq |n - 2|^p \int_{\mathbb{R}^n} |x|^{p-n} |\nabla(\Delta^{m-1} u)|^p dx \quad \forall u \in C_{c,r}^{2m}(\mathbb{R}^n \setminus \{0\}).$$

Theorem 1.2 and (1.5) provide explicit best constants in inequalities of the type

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx \geq c_{\alpha,k,j,p} \int_{\mathbb{R}^n} |x|^{\alpha-(k-j)p} |\nabla^j u|^p dx \quad \forall u \in C_{c,r}^k(\mathbb{R}^n \setminus \{0\}), \quad (1.10)$$

for any intermediate case  $j = 1, \dots, k - 1$ , see Remark 2.3.

Weighted high order inequalities for non radial functions are more involved. In [12] and [6], where  $p = 2$  and  $k = 2$  are assumed, it is proved that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^2 dx \geq \min_{i \in \mathbb{N} \cup \{0\}} |\gamma_{\alpha,2} + i(n - 2 + i)|^2 \int_{\mathbb{R}^n} |x|^{\alpha-4} |u|^2 dx$$

for any  $u \in C_c^2(\mathbb{R}^n \setminus \{0\})$ . In particular, the best constant vanishes if  $-\gamma_{\alpha,2}$  is an eigenvalue of the Laplace-Beltrami operator on the sphere. The problem of finding the best constant for weighted Rellich type inequalities in a non radial setting and for general parameters  $\alpha, k, p$  is still open.

Next we direct our attention to semilinear inequalities. Assume  $c_{\alpha,k,p} > 0$  and take an exponent  $q > p$ . For any integer  $j \in \{0, \dots, k-1\}$  let  $S_{k,q,j}(\alpha)$  to be the best constant in (1.3), that is,

$$S_{k,q,j}(\alpha) = \inf_{\substack{u \in \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx}{\left( \int_{\mathbb{R}^n} |x|^{-\beta_{k-j,q}} |\nabla^j u|^q dx \right)^{p/q}}. \quad (1.11)$$

In Theorem 7.14 of Section 7.3 we prove that  $S_{k,q,j}(\alpha)$  is positive and achieved in  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ . We refer also to Theorems 7.4 and 7.5 for shorter proofs in case  $k = 2$ .

If  $n > kp$ ,  $\alpha = 0$ ,  $j = 0$  and  $q = p^{k*} := \frac{np}{n-kp}$ , then  $\beta_{k,q} = 0$  and  $S_{k,q,0}(0)$  coincides with the radial Sobolev constant  $S_{k,p}^{k*}$ . Actually, (1.11) includes also the other  $(k-1)$  best constants  $S_{k,p}^*, S_{k,p}^{**}, \dots, S_{k,p}^{(k-1)*}$ , that are relative to the embeddings

$$\mathcal{D}_r^{k,p}(\mathbb{R}^n) \hookrightarrow \mathcal{D}_r^{j, \frac{np}{n-(k-j)p}}(\mathbb{R}^n), \quad j = 1, \dots, k-1.$$

We refer to Remarks 7.11 and 7.15 for details on this subject.

If  $k = 1$ ,  $\alpha > p - n$  and  $q > p$ , then the infimum  $S_{1,q,0}(\alpha)$  is closely related to the celebrated Caffarelli-Kohn-Nirenberg inequalities in [4]. In case  $p = 2$  its minimizers are explicitly known since the paper [9] by Catrina and Wang (see also [13]). In Theorem 7.2 we extend the Catrina-Wang uniqueness result to the non Hilbertian case  $p \neq 2$ . More precisely, we show that  $u \in \mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha dx) \setminus \{0\}$  solves

$$-\operatorname{div}(|x|^\alpha |\nabla u|^{p-2} \nabla u) = |x|^{-n+q \frac{n-p+\alpha}{p}} |u|^{q-2} u \quad \text{on } \mathbb{R}^n \quad (1.12)$$

if and only if  $u$  coincides with

$$U(|x|) = C \left( 1 + |x|^{\frac{(n-p+\alpha)(q-p)}{p(p-1)}} \right)^{\frac{p}{p-q}}$$

up to a change of sign and a rescaling, where  $C > 0$  is a computable constant. In particular,  $U$  achieves  $S_{1,q,0}(\alpha)$  and  $S_{1,q,0}(\alpha)$  is explicitly known. Notice that we do not need any a-priori assumption on the sign of  $u$ .

Now assume  $k = 2$  and  $\alpha \notin \{2p - n, np - n\}$ . Then  $c_{\alpha,2,p} > 0$  by Theorem 1.1 and the infima

$$S_{2,q,0}(\alpha) = \inf_{\substack{u \in \mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx}{\left( \int_{\mathbb{R}^n} |x|^{-n+q\frac{n-2p+\alpha}{p}} |u|^q dx \right)^{p/q}} \quad (1.13)$$

$$S_{2,q,1}(\alpha) := \inf_{\substack{u \in \mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx}{\left( \int_{\mathbb{R}^n} |x|^{-n+q\frac{n-2p+\alpha}{p}} |\nabla u|^q dx \right)^{p/q}} \quad (1.14)$$

are both positive and achieved. In Section 7.2 we explicitly provide the extremals for  $S_{2,q,1}(\alpha)$  and hence its value is computable in terms of Gamma functions, as in [3], [9] and [23].

In this introduction we limit ourselves to state a corollary of Theorem 7.5 that is concerned with the limiting embedding  $\mathcal{D}^{2,p}(\mathbb{R}^n) \hookrightarrow \mathcal{D}^{1,p^*}(\mathbb{R}^n)$ , where  $p > 2n$  and  $p^* = \frac{np}{n-p}$  is the (first order) critical exponent.

We denote by  $\Delta_{p^*} = \operatorname{div}(|\nabla \cdot|^{p^*-2} \nabla \cdot)$  the  $p^*$ -Laplace operator, and we identify functions that coincide up to a rescaling and a change of sign.

**Theorem 1.3** *Assume that  $n > 2p$ . Then the problem*

$$\Delta (|\Delta U|^{p-2} \Delta U) + \Delta_{p^*} U = 0 \quad (1.15)$$

*has a unique nontrivial solution  $U \in \mathcal{D}_r^{2,p}(\mathbb{R}^n)$ . More precisely,  $U$  achieves*

$$S_{2,p}^* := \inf_{\substack{u \in \mathcal{D}_r^{2,p}(\mathbb{R}^n) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |\Delta u|^p dx}{\left( \int_{\mathbb{R}^n} |\nabla u|^{p^*} dx \right)^{p/p^*}} \quad (1.16)$$

*and it is given by*

$$U(x) = \left( \frac{q(p-1)}{p} \left| \frac{n-p+\alpha}{p-1} \right|^p \right)^{\frac{1}{q-p}} \int_{|x|}^{\infty} s \left( 1 + s^{p^*} \right)^{\frac{p-n}{p}} ds.$$

There are a few ways to prove inequalities like (1.1) and (1.3). In [15], Mitidieri applied his powerful "simple approach" to compute  $c_{\alpha,k,p}$ , among other best

constants, when  $q = p$  and  $\gamma_{\alpha,k} \geq 0$ . Pointwise estimates are frequently used to obtain integral inequalities, see for instance the papers [4] by Caffarelli, Kohn and Nirenberg and the more recent [1], [2]. In presence of symmetries, Calanchi-Ruf in [5] and de Figueiredo-dos Santos-Miyagaki in [10] obtained embedding results as corollaries of a *radial lemma* (in the spirit of [18] and [14]).

Here we use a different approach. We start in Section 2 by proving Theorems 1.2 and 1.1 via the Hardy inequality for functions of one real variable. No pointwise estimates are needed. To study (1.3) we follow the opposite direction with respect to the above mentioned papers [5] and [10], that is, we first prove embeddings theorems, then we infer the desired inequalities. We focus our attention on (1.3) even if we can obtain also pointwise estimates, radial lemmas and informations on the regularity of radial functions, see Remark 7.12. Roughly speaking, to get more inequalities in case  $c_{\alpha,k,p} > 0$  we define a  $k$ -th order Emden-Fowler transform  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \rightarrow W^{k,p}(\mathbb{R})$  and we show that the induced norm on  $W^{k,p}(\mathbb{R})$  is equivalent to the standard one. Then, classical results about the space  $W^{k,p}(\mathbb{R})$  provide embedding theorems for  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ , and the inequalities we are interested in readily follow.

The first step in this program consists in highlighting a suitable class of equivalent norms on the Sobolev spaces  $W^{k,p}(\mathbb{R})$ . We will start with the lowest indexes  $k = 1$  and  $k = 2$  in Sections 4 and 5, respectively. The higher order case  $k \geq 3$  will be briefly discussed in Section 6.

**Notation.** We denote by  $c$  any nonnegative ininfluent constant.

We set  $\mathbb{R}_+ = (0, \infty)$ . For any integer  $n \geq 2$  we denote by  $\omega_n$  the  $n - 1$  dimensional measure of the unit sphere  $\mathbb{S}^{n-1}$  in  $\mathbb{R}^n$ .

If  $\Omega \subseteq \mathbb{R}^n$  is a rotationally invariant domain and  $k \geq 0$  is an integer, we denote by  $C_{c,r}^k(\Omega)$  the space of radially symmetric functions  $u \in C_c^k(\Omega)$ . For  $u \in C_{c,r}^1(\Omega)$  we let  $u'$  to be the radial derivative of  $u$ . Thus  $|x|u'(x) = \nabla u(x) \cdot x$ .

The exponent  $q' = \frac{q}{q-1}$  is conjugate exponent to  $q \in (1, \infty)$ .

Let  $\omega$  be a non-negative measurable function on a domain  $\Omega \in \mathbb{R}^n$ ,  $n \geq 1$ . The weighted Lebesgue space  $L^q(\Omega; \omega(x) dx)$  is the space of measurable maps  $u$  in  $\Omega$  with finite norm  $(\int_\Omega |u|^q \omega(x) dx)^{1/q}$ . For  $\omega \equiv 1$  we denote by  $\|u\|_q$  the standard norm in  $L^q(\Omega) = L^q(\Omega; dx)$ .

The norm in the Sobolev space  $W^{k,q}(\mathbb{R})$  is given by

$$\|g\|_{W^{k,q}} = \left( \int_{\mathbb{R}} |g^{(k)}|^q ds + \int_{\mathbb{R}} |g|^q ds \right)^{1/q}.$$

If  $n > kp$  then the space  $\mathcal{D}^{k,p}(\mathbb{R}^n)$  is the closure of  $C_c^\infty(\mathbb{R}^n)$  with respect to the norm

$$\|u\| = \left( \int_{\mathbb{R}^n} |\nabla^k u|^p dx \right)^{1/p}.$$

We put  $\mathcal{D}_r^{k,p}(\mathbb{R}^n) = \{u \in \mathcal{D}^{k,p}(\mathbb{R}^n) \mid u = u(|x|)\}$ .

Let  $p^{k*} = \frac{np}{n-kp}$  be the  $k$ -th order critical exponent. The radial Sobolev constant

$$S_{k,p}^{k*} := \inf_{\substack{u \in \mathcal{D}_r^{k,p}(\mathbb{R}^n) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |\nabla^k u|^p dx}{\left( \int_{\mathbb{R}^n} |u|^{p^{k*}} dx \right)^{p/p^{k*}}}$$

is positive and achieved (see also Theorem 7.14 and Remark 7.15 in Section 7.2 below).

Assume that  $k = 1$  or  $p = 2$ . Then it is well known that  $S_{k,p}^{k*}$  is the best constant in the embedding  $\mathcal{D}^{k,p}(\mathbb{R}^n) \hookrightarrow L^{p^{k*}}(\mathbb{R}^n)$ , that is,

$$S_{k,p}^{k*} = \inf_{\substack{u \in \mathcal{D}^{k,p}(\mathbb{R}^n) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |\nabla^k u|^p dx}{\left( \int_{\mathbb{R}^n} |u|^{p^{k*}} dx \right)^{p/p^{k*}}},$$

see [3], [23] and for instance [11] for the poliharmonic case.

The notation  $k*$  means  $\underbrace{* * \dots *}_{k \text{ times}}$ . If  $k \in \{1, 2\}$  we write  $*$ ,  $**$  instead of  $1*$ ,  $2*$ , respectively.

## 2 Higher order Rellich inequalities

Throughout this paper we will use the Hardy inequality for functions in  $C_c^1(\mathbb{R}_+)$  several times. We recall that

$$\int_0^\infty r^a |\omega'|^p dr \geq \left| \frac{a+1-p}{p} \right|^p \int_0^\infty r^{a-p} |\omega|^p dr \quad (2.1)$$

for any  $a \in \mathbb{R}$ ,  $p > 1$  and  $\omega \in C_c^1(\mathbb{R}_+)$ . Moreover, the constant in the right hand side is sharp and not achieved.

We point out a simple but very useful corollary to the Hardy inequality (2.1).

**Lemma 2.1** *Let  $\tau, \lambda \in \mathbb{R}$ ,  $p > 1$  and  $v \in C_c^2(\mathbb{R}_+)$ . Then the inequalities*

$$\int_0^\infty r^\tau |v'' + (\lambda - 1)r^{-1}v'|^p dr \geq \left| \frac{\tau + 1 - \lambda p}{p} \right|^p \int_0^\infty r^{\tau-p} |v'|^p dr \quad (2.2)$$

$$\int_0^\infty r^\tau |v'' + (\lambda - 1)r^{-1}v'|^p dr \geq \left| \frac{(\tau + 1 - \lambda p)(\tau + 1 - 2p)}{p^2} \right|^p \int_0^\infty r^{\tau-2p} |v|^p dr \quad (2.3)$$

*hold with sharp constants.*

**Proof.** We start by noticing that (2.3) follows from (2.2), thanks to the Hardy inequality (2.1). To prove (2.2) we use again (2.1) to estimate

$$\begin{aligned} \int_0^\infty r^\tau |v'' + (\lambda - 1)r^{-1}v'|^p dr &= \int_0^\infty r^\tau \left| r^{1-\lambda} (r^{\lambda-1}v')' \right|^p dr \\ &= \int_0^\infty r^{\tau-(\lambda-1)p} \left| (r^{\lambda-1}v')' \right|^p dr \\ &\geq \left| \frac{\tau + 1 - \lambda p}{p} \right|^p \int_0^\infty r^{\tau-\lambda p} |r^{\lambda-1}v'|^p dr. \end{aligned}$$

Thus (2.2) holds, and the proof is complete.  $\square$

## 2.1 Proof of Theorem 1.2.

Fix any  $u \in C_{c,r}^{2m}(\mathbb{R}^n \setminus \{0\})$ . Since

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx &= \omega_n \int_0^\infty r^{n-1+\alpha} |u'' + (n-1)r^{-1}u'|^p dr \\ \int_{\mathbb{R}^n} |x|^{\alpha-p} |\nabla u|^p dx &= \omega_n \int_0^\infty r^{n-1+\alpha-p} |u'|^p dr, \end{aligned}$$

then from (2.2) it follows that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx \geq \left| n - \frac{n+\alpha}{p} \right|^p \int_{\mathbb{R}^n} |x|^{\alpha-p} |\nabla u|^p dx. \quad (2.4)$$

Thus (1.9) is proved in case  $m = 1$ . If  $m > 1$  it suffices to write down (2.4) with  $\Delta^{m-1}u \in C_{c,r}^2(\mathbb{R}^n \setminus \{0\})$  instead of  $u$ .  $\square$

## 2.2 Proof of Theorem 1.1

We first prove (1.7) in case  $k = 2m$  is an even integer. We have to show that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta^m u|^p dx \geq \prod_{h=1}^m |\gamma_{\alpha, 2h}|^p \int_{\mathbb{R}^n} |x|^{\alpha-2mp} |u|^p dx \quad \forall u \in C_{c,r}^{2m}(\mathbb{R}^n \setminus \{0\}). \quad (2.5)$$

If  $m = 1$  then (2.5) reduces to (1.8), that is an immediate consequence of (2.4) and of the Hardy inequality (1.5). Assume that (2.5) holds for some  $m \geq 1$  and for any  $\alpha \in \mathbb{R}$ . Fix  $u \in C_{c,r}^{2m+2}(\mathbb{R}^n \setminus \{0\})$  and use (1.8) to infer

$$\int_{\mathbb{R}^n} |x|^{\alpha-2mp} |\Delta u|^p dx \geq |\gamma_{\alpha, 2m+2}|^p \int_{\mathbb{R}^n} |x|^{\alpha-2(m+1)p} |u|^p dx,$$



since  $\gamma_{\alpha-2mp,2} = \gamma_{\alpha,2m+2}$ . Thus we have that

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^\alpha |\Delta^{m+1} u|^p dx &= \int_{\mathbb{R}^n} |x|^\alpha |\Delta^m(\Delta u)|^p dx \\ &\geq \prod_{h=1}^m |\gamma_{\alpha,2h}|^p \int_{\mathbb{R}^n} |x|^{\alpha-2mp} |\Delta u|^p dx \\ &\geq \left( \prod_{h=1}^m |\gamma_{\alpha,2h}|^p \right) |\gamma_{2m+2}|^p \int_{\mathbb{R}^n} |x|^{\alpha-2(m+1)p} |u|^p dx, \end{aligned}$$

as desired. If  $k = 2m + 1$  is odd we use the Hardy inequality and the first part of the proof to get

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^\alpha |\nabla(\Delta^m u)|^p dx &\geq |H_\alpha|^p \int_{\mathbb{R}^n} |x|^{\alpha-p} |\Delta^m u|^p dx \\ &\geq |H_\alpha|^p \left( \prod_{h=1}^m |\gamma_{\alpha-p,2h}|^p \right) \int_{\mathbb{R}^n} |x|^{\alpha-p} |\Delta^m u|^p dx \end{aligned}$$

for any  $u \in C_{c,r}^{2m+1}(\mathbb{R}^n \setminus \{0\})$ . The conclusion readily follows, as  $\gamma_{\alpha-p,2h} = \gamma_{\alpha,2h+1}$  for any integer  $h$   $\square$

**Remark 2.2** *An alternative proof of Theorem 1.1 is suggested in Remark 7.10.*

**Remark 2.3** *Tedious computations allow us to find the best constant  $c_{\alpha,k,j,p}$  in (1.10). Theorem 1.2 and the Hardy inequality provide the values of  $c_{\alpha,k,j,p}$  for  $j = k - 1$  and  $j = k - 2$ . For smaller indexes we put  $\delta_j = n - 1 - \frac{n+\alpha}{p} + k - j$ . If  $k = 2m$  is even we have*

$$c_{\alpha,2m,j,p} = \begin{cases} \prod_{h=1}^{m-i} |\gamma_{\alpha,2h}|^p & \text{if } j = 2i \\ |\delta_j|^p \prod_{h=1}^{m-i-1} |\gamma_{\alpha,2h}|^p & \text{if } j = 2i + 1, \end{cases}$$

while if  $k = 2m + 1$  it results that

$$c_{\alpha,(2m+1),j,p} = \begin{cases} |H_\alpha|^p \prod_{h=1}^{m-i} |\gamma_{\alpha,2h+1}|^p & \text{if } j = 2i, \\ |H_\alpha \delta_j|^p \prod_{h=1}^{m-i-1} |\gamma_{2h+1}|^p & \text{if } j = 2i + 1. \end{cases}$$

### 3 The $k$ -th order Emden-Fowler transform

Let  $p > 1$ ,  $\alpha \in \mathbb{R}$  and let  $k \geq 1$  be a given integer. From now on we assume that

$$\begin{cases} \gamma_{\alpha,h} \neq 0 & \forall h = 1, \dots, m & \text{if } k = 2m \text{ is even,} \\ \gamma_{\alpha,h} \neq 0 & \forall h = 1, \dots, m \text{ and } H_\alpha \neq 0 & \text{if } k = 2m + 1 \text{ is odd.} \end{cases} \quad (3.1)$$

Let us briefly describe our strategy. Thanks to Theorem 1.1, we can define the space  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  as the completion of  $C_{c,r}^k(\mathbb{R}^n \setminus \{0\})$  with respect to the norm in (1.2). Next we define the (inverse)  $k$ -th order Emden-Fowler transform

$$\mathcal{T}_k : C_c^k(\mathbb{R}) \rightarrow C_{c,r}^k(\mathbb{R}^n \setminus \{0\}) \quad (\mathcal{T}_k g)(x) = |x|^{-H_k} g(-\log |x|),$$

where we have set

$$H_{\alpha,k} = \frac{n + \alpha}{p} - k. \quad (3.2)$$

Notice that  $H_{\alpha,1} = H_\alpha$ , compare with (1.6) and (1.5). Then we show that  $\mathcal{T}_k$  extends to a bicontinuous isomorphism  $W^{k,p}(\mathbb{R}) \rightarrow \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ . Finally, from the Sobolev embeddings of  $W^{k,p}(\mathbb{R})$  we readily infer all the inclusions we need to prove inequalities of the type (1.3).

The first step in this program consists in highlighting suitable classes of equivalent norms in  $W^{k,p}(\mathbb{R})$ , starting from the lowest order  $k = 1$ .

### 4 Equivalent norms on $W^{1,p}(\mathbb{R})$

We point out a simple lemma, based on the Hardy inequality for functions in  $C_c^1(\mathbb{R}_+)$ .

**Lemma 4.1** *Let  $p > 1$  and  $\lambda \in \mathbb{R}$ . Then*

$$M_p(\lambda) := \inf_{\substack{f \in W^{1,p}(\mathbb{R}) \\ f \neq 0}} \frac{\int_{\mathbb{R}} |f' - \lambda f|^p ds}{\int_{\mathbb{R}} |f|^p ds} = |\lambda|^p$$

*and  $M_p(\lambda)$  is not achieved.*

**Proof.** To any  $f \in C_c^1(\mathbb{R})$  we associate the function  $v \in C_c^1(\mathbb{R}_+)$  defined by  $v(r) := r^\lambda f(-\log r)$ . Simple computations show that

$$\frac{\int_{\mathbb{R}} |f' - \lambda f|^p ds}{\int_{\mathbb{R}} |f|^p ds} = \frac{\int_0^\infty r^{(1-\lambda)p-1} |v'|^p dr}{\int_0^\infty r^{-\lambda p-1} |v|^p dr},$$

and the conclusion readily follows by using (2.1) and a density argument.  $\square$

The next proposition furnishes the equivalent norms we need in case  $k = 1$ . Its proof is immediate, by Lemma 4.1 and by Sobolev embedding theorem.

**Proposition 4.2** *Let  $p > 1$  and  $\lambda \in \mathbb{R} \setminus \{0\}$ . Then*

$$\|f\| := \left( \int_{\mathbb{R}} |f' - \lambda f|^p ds \right)^{1/p}$$

*is equivalent to the standard norm on  $W^{1,p}(\mathbb{R})$ . Thus, for any  $q > p$  the infimum*

$$M_{p,q}(\lambda) := \inf_{\substack{f \in W^{1,p}(\mathbb{R}) \\ f \neq 0}} \frac{\int_{\mathbb{R}} |f' - \lambda f|^p ds}{\left( \int_{\mathbb{R}} |f|^q ds \right)^{p/q}} \quad (4.1)$$

*is positive.*

If  $p = 2 < q$ , then extremals for

$$M_{2,q}(\lambda) = \inf_{\substack{f \in H^1(\mathbb{R}) \\ f \neq 0}} \frac{\int_{\mathbb{R}} |f' - \lambda f|^2 ds}{\left( \int_{\mathbb{R}} |f|^q ds \right)^{2/q}} = \inf_{\substack{f \in H^1(\mathbb{R}) \\ f \neq 0}} \frac{\int_{\mathbb{R}} (|f'|^2 + \lambda^2 |f|^2) ds}{\left( \int_{\mathbb{R}} |f|^q ds \right)^{2/q}}$$

give rise to nontrivial solutions of the Emden-Fowler (or Schrödinger) equation

$$-f'' + \lambda^2 f = |f|^{q-2} f \quad \text{on } \mathbb{R}. \quad (4.2)$$

It has been shown in [9] that, up to translations, equation (4.2) has a unique positive solution  $F \in H^1(\mathbb{R})$ , which is explicitly known. The interest of Catrina and Wang in the ODE (4.2) was motivated by its relevance with the Caffarelli-Kohn-Nirenberg inequalities in the Hilbertian case  $p = 2$ .

**Remark 4.3** *The minimization problem in (4.1) is non compact, due to translations in  $\mathbb{R}$ . By nowadays standard arguments one can prove that for every bounded minimizing sequence  $f_h$ , there exists a sequence  $s_h$  in  $\mathbb{R}$  such that  $f_h(\cdot - s_h)$  is relatively compact in  $W^{1,p}(\mathbb{R})$ . Hence,  $M_{p,q}(\lambda)$  is attained by some function  $f \neq 0$  which solves*

$$- (|f' - \lambda f|^{p-2}(f' - \lambda f))' - \lambda |f' - \lambda f|^{p-2}(f' - \lambda f) = |f|^{q-2}f \quad \text{on } \mathbb{R} \quad (4.3)$$

*up to a Lagrange multiplier.*

Now we state a uniqueness result for nontrivial solutions  $f \in W^{1,p}(\mathbb{R})$  to (4.3). Notice that we do not require any sign assumption on  $f$ . Thus some care is needed, as the exponents  $p, q$  might be smaller than 2.

We identify functions that coincide up to a translation and a change of sign.

**Theorem 4.4** *Let  $p > 1$ ,  $q > p$  and  $\lambda \in \mathbb{R} \setminus \{0\}$ . Then the ordinary differential equation (4.3) has a unique nontrivial solution  $F \in W^{1,p}(\mathbb{R})$ . More precisely,  $F$  achieves the best constant  $M_{p,q}(\lambda)$ , and it is given by*

$$F(s) = \left( q \left( \frac{p}{p-1} \right)^{p-1} \left| \frac{\lambda}{2} \right|^p \right)^{\frac{1}{q-p}} e^{\frac{\lambda(p-2)}{2(p-1)}s} \left( \cosh \left( \frac{\lambda(q-p)}{2(p-1)} s \right) \right)^{\frac{p}{p-q}}. \quad (4.4)$$

**Proof.** For any nontrivial solution  $f \in W^{1,p}(\mathbb{R})$  to (4.3) we define

$$\varphi = |f' - \lambda f|^{p-2}(f' - \lambda f).$$

The pair  $f, \varphi$  solves

$$\begin{cases} f' - \lambda f = |\varphi|^{p'-2}\varphi \\ -\varphi' - \lambda\varphi = |f|^{q-2}f \end{cases} \quad (4.5)$$

in the sense of distributions. Notice that  $p', q$  satisfy the standard anticoercivity assumption  $(p' - 1)(q - 1) > 1$ . Clearly,  $\varphi \in L^{p'}(\mathbb{R})$  and  $\|\varphi\|_{p'} = \|f' - \lambda f\|_p$ . Since  $-\varphi' = \lambda\varphi + |f|^{q-2}f \in L^{p'}(\mathbb{R})$  by Sobolev embeddings, then  $\varphi \in W^{1,p'}(\mathbb{R})$ . Thus  $\varphi \in C^1(\mathbb{R})$ , as  $f$  and  $\varphi$  are continuous function. But then also  $f$  is of class  $C^1$ , since  $f' = \lambda f + |\varphi|^{p'-2}\varphi$ . Thus the pair  $f, \varphi$  is a classical homoclinic solution to (4.5).

The system (4.5) is conservative, with with Hamiltonian energy

$$H(f, \varphi) = \lambda f \varphi + \frac{1}{q} |f|^q + \frac{1}{p'} |\varphi|^{p'}.$$

In particular, (4.5) is equivalent to

$$\begin{cases} f' = \partial_2 H(f, g) \\ g' = -\partial_1 H(f, g). \end{cases} \quad (4.6)$$

From  $f \in W^{1,p}(\mathbb{R})$ ,  $\varphi \in W^{1,p'}(\mathbb{R})$  one infers that  $f, \varphi$  vanish at infinity, and therefore

$$\lambda f \varphi + \frac{1}{q} |f|^q + \frac{1}{p'} |\varphi|^{p'} = 0. \quad (4.7)$$

Notice that  $\lambda f \varphi < 0$  on the set  $\{f \neq 0\} = \{\varphi \neq 0\}$ . We can assume that  $f$  achieves its positive maximum at some point  $s_0$ . Using  $f'(s_0) = 0$ , (4.5) and (4.7) one can uniquely compute the values of  $f(s_0) > 0$  and  $\lambda \varphi(s_0) < 0$ . Since for any initial datum  $f_0 > 0$ ,  $\varphi_0 \neq 0$  the Cauchy problem for (4.6) has a unique local solution, to conclude the proof we only have to show that the pair  $F, \Phi$  solves (4.5), where  $\Phi$  is given by

$$\Phi = |F' - \lambda F|^{p-2} (F' - \lambda F).$$

In order to avoid long computations one can argue as follows. Put

$$k = \left( q \left( \frac{p}{p-1} \right)^{p-1} \left| \frac{\lambda}{2} \right|^p \right)^{\frac{1}{q-p}}, \quad c_1 = \frac{\lambda(p-2)}{2(p-1)}, \quad c_2 = \frac{\lambda(q-p)}{2(p-1)},$$

so that  $F(s) = k e^{c_1 s} (\cosh c_2 s)^{\frac{p}{p-q}}$ , and compute

$$\Phi = - \left( k \frac{p}{2(p-1)} \right)^{p-1} |\lambda|^{p-2} \lambda e^{(p-1)(c_1+c_2)s} (\cosh c_2 s)^{\frac{q(p-1)}{p-q}}.$$

Now it is easy to check that the pair  $F, \Phi$  satisfies the conservation law (4.7), that is sufficient to conclude that  $F, \Phi$  solves (4.5), as desired.  $\square$

## 5 Equivalent norms on $W^{2,p}(\mathbb{R})$

We start with a preliminary result.

**Lemma 5.1** *Let  $p > 1$  and  $A, \gamma \in \mathbb{R}$  with  $A^2 + \gamma \geq 0$ . Then*

$$I_p(A, \gamma) := \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |g'' - 2A g' - \gamma g|^p ds}{\int_{\mathbb{R}} |g|^p ds} = |\gamma|^p$$

*and  $I_p(A, \gamma)$  is not achieved.*

**Proof.** We introduce the constants

$$\lambda = 2 + 2\sqrt{A^2 + \gamma}, \quad \tau = p \frac{\lambda + 2 - 2A}{2} - 1.$$

To any  $g \in C_c^2(\mathbb{R})$  we associate the function  $v(r) := r^{2 - \frac{\tau+1}{p}} g(-\log r)$ . Notice that

$$\begin{aligned} \int_{\mathbb{R}} |g'' - A g' - \gamma g|^p ds &= \int_0^\infty r^\tau |v'' + (\lambda - 1)r^{-1}v'|^p dr, \\ \int_{\mathbb{R}} |g|^p ds &= \int_0^\infty r^{\tau-2p} |v|^p dr. \end{aligned}$$

Therefore, by a density argument,

$$I_p(A, \gamma) = \inf_{\substack{v \in C_c^2(\mathbb{R}_+) \\ v \neq 0}} \frac{\int_0^\infty r^\tau |v'' + (\lambda - 1)r^{-1}v'|^p dr}{\int_0^\infty r^{\tau-2p} |v|^p dr}.$$

Since  $(\tau + 1 - \lambda p)(\tau + 1 - 2p) = -\gamma p^2$ , the conclusion follows by Lemma 2.1.  $\square$

**Proposition 5.2** *Let  $p > 1$  and  $A, \gamma \in \mathbb{R}$  with  $A^2 + \gamma \geq 0$ . If  $\gamma \neq 0$  then*

$$\|g\|_{A, \gamma} := \left( \int_{\mathbb{R}} |g'' - 2A g' - \gamma g| ds \right)^{1/p}$$

*is an equivalent norm on  $W^{2,p}(\mathbb{R})$ . Moreover, for any  $q > p$  the infimum*

$$I_{p,q}(A, \gamma) := \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |g'' - 2A g' - \gamma g|^p ds}{\left( \int_{\mathbb{R}} |g|^q ds \right)^{p/q}}$$

*is positive and achieved in  $W^{2,p}(\mathbb{R})$ .*

**Proof.** Fix a small  $\varepsilon > 0$  such that  $|2A|\varepsilon \leq 1/2$ . We recall that there exists a constant  $C_\varepsilon > 0$  such that  $\|g'\|_p \leq \varepsilon\|g''\|_p + C_\varepsilon\|g\|_p$  for any  $g \in W^{2,p}(\mathbb{R})$ . Using also Lemma 5.1 we find that

$$\begin{aligned}\|g\|_{W^{2,p}} &\leq 2 \left(1 + \frac{|2A|C_\varepsilon + |\gamma| + 1}{|\gamma|^p}\right) \|g'' - 2A g' - \gamma g\|_p \\ \|g'' - 2A g' - \gamma g\|_p &\leq (2 + |2A|C_\varepsilon + |\gamma|) \|g\|_{W^{2,p}}.\end{aligned}$$

Thus the norm  $\|\cdot\|_{A,\gamma}$  is equivalent to the standard one.

Since  $W^{2,p}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$  by Sobolev embedding, then  $I_{p,q}(A, \gamma) > 0$  by the first part of the proof. By nowadays standard arguments, it is easy to prove that every bounded minimizing sequence for  $I_{p,q}(A, \gamma)$  is relatively compact in  $W^{2,p}(\mathbb{R})$  up to translations in  $\mathbb{R}$ . In particular,  $I_{p,q}(A, \gamma)$  is attained in  $W^{2,p}(\mathbb{R})$ .  $\square$

Now we focus our attention on the inclusions  $W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,q}(\mathbb{R})$ , where  $q \geq p$ . We start with the "linear" case  $q = p$ .

**Lemma 5.3** *Let  $p > 1$ ,  $A, \gamma, B \in \mathbb{R}$  with  $A^2 + \gamma \geq 0$ . Let*

$$J_p(A, \gamma, H) := \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |g'' - 2A g' - \gamma g|^p ds}{\int_{\mathbb{R}} |g' + Hg|^p ds}.$$

i) *If  $\gamma \neq 0$  then  $J_p(A, \gamma, H) > 0$ .*

ii)  *$J_p(A, 0, 0) = |2A|^p$ .*

iii) *If  $H \neq 0$  and  $H^2 + 2AH - \gamma = 0$ , then  $J_p(A, \gamma, H) = \left|\frac{\gamma}{H}\right|^p$ .*

**Proof.** If  $\gamma \neq 0$  then  $\|g\|_{A,\gamma}$  is an equivalent norm on  $W^{2,p}(\mathbb{R})$  by Proposition 5.2. Hence  $J_p(A, \gamma, H) > 0$ , since  $W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,p}(\mathbb{R})$ .

To check ii) one can reproduce the trick in the proof of Lemma 5.1, or can argue as follows. First notice that  $J_p(A, 0, 0) \geq |2A|^p$  by Lemma 4.1. To prove the opposite inequality use a rescaling argument. Take any  $g \in C_c^2(\mathbb{R}) \setminus \{0\}$  and test  $J_p(A, 0, 0)$  with  $s \mapsto g(ts)$ , where  $t \rightarrow 0^+$ . The conclusion is readily achieved, as

$$J_p(A, 0, 0) \leq \frac{\int_{\mathbb{R}} |tg'' - 2A g'|^p ds}{\int_{\mathbb{R}} |g'|^p ds} = |2A| + o(1).$$

It remains to check *iii*). Notice that

$$J_p(A, \gamma, H) = \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} \left| (g' + Hg)' - \frac{\gamma}{H} (g' + Hg) \right|^p ds}{\int_{\mathbb{R}} |g' + Hg|^p ds} \geq \left| \frac{\gamma}{H} \right|^p$$

by Lemma 4.1. Then use rescaling as before to prove the opposite inequality.  $\square$

Now we direct our attention to "semilinear" inequalities. For any  $q > p$ ,  $\lambda \in \mathbb{R}$ , the infimum  $M_{p,q}(\lambda)$  has been defined in (4.1).

**Proposition 5.4** *Let  $p > 1$ ,  $q > p$ ,  $A, \gamma, H \in \mathbb{R}$  with  $A^2 + \gamma \geq 0$ , and put*

$$J_{p,q}(A, \gamma, H) := \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |g'' - 2A g' - \gamma g|^p ds}{\left( \int_{\mathbb{R}} |g' + Hg|^q ds \right)^{p/q}}.$$

- i) If  $\gamma \neq 0$  then  $J_{p,q}(A, \gamma, H)$  is positive and it is achieved in  $W^{2,p}(\mathbb{R})$ .*
- ii)  $J_{p,q}(A, 0, -2A) = 0$  for any  $A \in \mathbb{R}$ .*
- iii)  $J_{p,q}(A, 0, 0) = M_{p,q}(2A)$  and it is not achieved.*

**Proof.** If  $\gamma \neq 0$  then  $\|g\|_{A,\gamma}$  is an equivalent norm on  $W^{2,p}(\mathbb{R})$  by Proposition 5.2. Thus *i*) readily follows, as  $W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,q}(\mathbb{R})$ . To prove that  $J_{p,q}(A, \gamma, H)$  is attained use standard arguments in translation-invariant problems, as for Proposition 5.2.

The fact that  $J_{p,q}(A, 0, -2A) = 0$  can be proved via rescaling, since

$$J_{p,q}(A, 0, -2A) = \inf_{\substack{g \in W^{2,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |(g' - 2A g)'|^p ds}{\left( \int_{\mathbb{R}} |g' - 2A g|^q ds \right)^{p/q}}.$$

To check *iii*) we first notice that

$$J_{p,q}(A, 0, 0) \geq \inf_{\substack{f \in W^{1,p}(\mathbb{R}) \\ f \neq 0}} \frac{\int_{\mathbb{R}} |f' - 2A f|^p ds}{\left( \int_{\mathbb{R}} |f|^q ds \right)^{p/q}} = M_{p,q}(2A).$$



Next, for any function  $f \in C_c^1(\mathbb{R})$ ,  $f \neq 0$  we put  $g(s) = \int_{-\infty}^s f(t) dt$ . Then  $g$  is bounded,  $g(s) \equiv 0$  for  $s \ll 0$  and  $g(s)$  is a constant for  $s \gg 0$ , so that in general  $g \notin L^p(\mathbb{R})$ . Take a function  $\eta \in C^2(\mathbb{R})$ , such that  $0 \leq \eta \leq 1$ ,  $\eta \equiv 1$  on  $(-\infty, 1)$  and  $\eta \equiv 0$  on  $(2, \infty)$ . We test  $J_{p,q}(A, 0, 0)$  with the function  $g_h(s) = \eta(h^{-1}s)g(s)$ , where  $h \geq 1$  is an integer. Notice that  $g_h \in W^{2,p}(\mathbb{R})$  since it is smooth and it has compact support. It is not difficult to show that  $g'_h \rightarrow g' = f$  in  $L^p(\mathbb{R})$  and in  $L^q(\mathbb{R})$ ,  $g''_h \rightarrow g'' = f'$  in  $L^p(\mathbb{R})$ . Thus

$$J_{p,q}(A, 0, 0) \leq \frac{\int_{\mathbb{R}} |g''_h - 2A g'_h|^p ds}{\left( \int_{\mathbb{R}} |g'_h|^q ds \right)^{p/q}} = \frac{\int_{\mathbb{R}} |f' - 2A f|^p ds}{\left( \int_{\mathbb{R}} |f|^q ds \right)^{p/q}} + o(1).$$

Thus  $J_{p,q}(A, 0, 0) = M_{p,q}(2A)$ , as  $f$  was arbitrarily chosen. It remains to check that  $J_{p,q}(A, 0, 0)$  is not attained. Assume that  $g \in W_{\text{loc}}^{2,p}(\mathbb{R})$  is a non constant function such that  $g' \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$ ,  $g'' \in L^p(\mathbb{R})$  and

$$\int_{\mathbb{R}} |g'' - 2A g'|^p ds = J_{p,q}(A, 0, 0) \left( \int_{\mathbb{R}} |g'|^q ds \right)^{p/q} = M_{p,q}(2A) \left( \int_{\mathbb{R}} |g'|^q ds \right)^{p/q}. \quad (5.1)$$

Then  $g' \in W^{1,p}(\mathbb{R})$  achieves  $M_{p,q}(2A)$ , and hence  $g'$  has constant sign (use a standard convexity argument or Theorem 4.4). In particular  $g$  is monotone, that implies that  $g \notin L^p(\mathbb{R})$ . Thus  $g$  does not achieve  $J_{p,q}(A, 0, 0)$ .  $\square$

**Remark 5.5** If  $A \neq 0$  then  $J_{p,q}(A, 0, 0) = M_{p,q}(2A) > 0$  by Proposition 4.2. In Theorem 4.4 we proved that the infimum  $M_{p,q}(2A)$  is achieved by a unique and positive function  $F \in W^{1,p}(\mathbb{R})$ . Therefore, any primitive  $g$  of  $F$  satisfies  $g', g'' \in L^p(\mathbb{R})$  and (5.1). However,  $g \notin W^{2,p}(\mathbb{R})$  since  $g$  is increasing on  $\mathbb{R}$ .

In order to simplify notations we introduce the differential operators

$$\begin{aligned} \mathcal{B}_+ \eta &= \eta' + H\eta, & \mathcal{B}_- \eta &= \eta' - H\eta, \\ \mathcal{L}_+ \eta &= -\eta'' + 2A\eta' + \gamma\eta, & \mathcal{L}_- \eta &= -\eta'' - 2A\eta' + \gamma\eta. \end{aligned}$$

**Remark 5.6** Assume  $\gamma \neq 0$ . Then any minimizer  $g \in W^{2,p}$  for  $J_{p,q}(A, \gamma, H)$  is, up to a Lagrange multiplier, a weak solution to the fourth order differential equation

$$\mathcal{L}_- (|\mathcal{L}_+ g|^{p-2} \mathcal{L}_+ g) = -\mathcal{B}_- (|\mathcal{B}_+ g|^{q-2} \mathcal{B}_+ g) \quad \text{on } \mathbb{R}. \quad (5.2)$$

In the next result we identify functions that coincide up to multiplicative constants and composition with translations in  $\mathbb{R}$ .

**Theorem 5.7** *Let  $p > 1$ ,  $q > p$ ,  $A, H, \gamma \in \mathbb{R}$  with  $\gamma, H \neq 0$ ,  $A^2 + \gamma \geq 0$  and*

$$H^2 + 2AH - \gamma = 0. \quad (5.3)$$

*Then (5.2) has a unique nontrivial solution  $G \in W^{2,p}(\mathbb{R})$ . More precisely,  $G$  achieves the best constant  $J_{p,q}(A, \gamma, H)$ , and  $J_{p,q}(A, \gamma, H) = M_{p,q}(\frac{\gamma}{H})$ ,*

$$\begin{aligned} G(s) &= k e^{-Hs} \int_{e^{-s}}^{\infty} t^{\frac{\gamma}{H(p-1)} - H - 1} \left(1 + t^{\frac{\gamma(q-p)}{H(p-1)}}\right)^{\frac{p}{p-q}} dt \quad \text{if } H > 0 \\ G(s) &= k e^{-Hs} \int_0^{e^{-s}} t^{\frac{\gamma}{H(p-1)} - H - 1} \left(1 + t^{\frac{\gamma(q-p)}{H(p-1)}}\right)^{\frac{p}{p-q}} dt \quad \text{if } H < 0, \end{aligned}$$

where

$$k = \left( q \left( \frac{p}{p-1} \right)^{p-1} \left| \frac{\gamma}{2H} \right|^p \right)^{\frac{1}{q-p}}.$$

**Proof.** First of all one has to prove that  $G$  is a  $W^{2,p}(\mathbb{R})$ -solution to (5.2). We indicate here a way to minimize computations. We notice that

$$G(s) = \begin{cases} e^{-Hs} \int_{-\infty}^s e^{Ht} F(t) dt & \text{if } H > 0 \\ e^{-Hs} \int_s^{\infty} e^{Ht} F(t) dt & \text{if } H < 0, \end{cases}$$

where  $F \in W^{1,p}(\mathbb{R})$  is the function defined in (4.4) with  $\lambda = \frac{\gamma}{H}$ . Thus by Theorem 4.4 we know that  $F = G' + HG$  achieves the infimum  $M_{p,q}(\frac{\gamma}{H})$  and solves

$$- \left( |f' - \frac{\gamma}{H} f|^{p-2} (f' - \frac{\gamma}{H} f) \right)' - \frac{\gamma}{H} |f' - \frac{\gamma}{H} f|^{p-2} (f' - \frac{\gamma}{H} f) = |f|^{q-2} f. \quad (5.4)$$

Since  $G$  decays exponentially at  $\pm\infty$ , then clearly  $G \in L^p(\mathbb{R})$ . Hence  $G \in W^{2,p}(\mathbb{R})$ , as  $G' = F - HG \in L^p(\mathbb{R})$ . Now we use (5.3) to get

$$F' - \frac{\gamma}{H} F = G'' - 2AG' - \gamma G = -\mathcal{L}_+ G.$$

Hence, we have showed that  $G$  solves

$$\begin{aligned} (|\mathcal{L}_+ G|^{p-2}(\mathcal{L}_+ G))' + \frac{\gamma}{H} |\mathcal{L}_+ G|^{p-2}(\mathcal{L}_+ G) &= |G' + HG|^{q-2}(G' + HG) \\ &= |\mathcal{B}_+ G|^{q-2} \mathcal{B}_+ G. \end{aligned} \quad (5.5)$$

Finally, we apply the operator  $-\mathcal{B}_-$  to both sides of (5.5) and we use again (5.3) to get that  $G$  is a solution to (5.2).

Now, assume that  $g \in W^{2,p}(\mathbb{R}) \setminus \{0\}$  solves (5.2), and put

$$f := g' + Hg = \mathcal{B}_+ g \in W^{1,p}(\mathbb{R}).$$

First of all we show that  $f$  solves (5.4). By (5.3) we have that

$$\varphi := \left| f' - \frac{\gamma}{H} f \right|^{p-2} \left( f' - \frac{\gamma}{H} f \right) = -|\mathcal{L}_+ g|^{p-2} \mathcal{L}_+ g \in L^{p'}(\mathbb{R})$$

with pointwise a.e. equalities. Thus  $\varphi$  is a distributional solution to

$$\mathcal{L}_- \varphi = \mathcal{B}_- (|f|^{q-2} f), \quad (5.6)$$

as  $g$  solves (5.2). Now  $|f|^{q-1} \in L^{p'}(\mathbb{R})$  as  $g \in W^{2,p}(\mathbb{R}) \hookrightarrow W^{1,\tau}(\mathbb{R})$  for any  $\tau \geq p$ , and therefore  $\varphi$  satisfies

$$\int_{\mathbb{R}} \varphi (\mathcal{L}_+ \eta) \, ds = \langle \mathcal{L}_- \varphi, \eta \rangle = \langle \mathcal{B}_- (|f|^{q-2} f), \eta \rangle = \int_{\mathbb{R}} |f|^{q-2} f (\mathcal{B}_+ \eta) \, ds$$

for any  $\eta \in C_c^\infty(\mathbb{R})$ . In particular we infer that

$$\left| \int_{\mathbb{R}} \varphi (\mathcal{L}_+ \eta) \, ds \right| \leq c \left( \int_{\mathbb{R}} |\mathcal{B}_+ \eta|^p \, ds \right)^{1/p}$$

where the constant  $c > 0$  depends only on  $g$ . Since  $H \neq 0$  by assumption, by Proposition 4.2 we get that the functional  $\eta \mapsto \int_{\mathbb{R}} \varphi (\mathcal{L}_+ \eta) \, ds$  is continuous with respect to the  $W^{1,p}(\mathbb{R})$  topology, that is,  $\varphi \in W^{1,p'}(\mathbb{R})$  and  $\varphi$  solves (5.6) in a weak sense. On the other hand, in the dual  $W^{-1,p}(\mathbb{R})$  we can compute

$$\mathcal{L}_- \varphi = -\varphi'' - 2A\varphi' + \gamma\varphi = -\left(\varphi' + \frac{\gamma}{H}\varphi\right)' + H\left(\varphi' + \frac{\gamma}{H}\varphi\right) = -\mathcal{B}_- \left(\varphi' + \frac{\gamma}{H}\varphi\right),$$

thanks to (5.3). Thus we have shown that

$$-\mathcal{B}_- \left(\varphi' + \frac{\gamma}{H}\varphi\right) = \mathcal{B}_- (|f|^{q-2} f).$$

The operator  $\mathcal{B}_- : W^{1,p}(\mathbb{R}) \rightarrow W^{-1,p}(\mathbb{R})$  is invertible, and therefore it holds that  $-\varphi' - \frac{\gamma}{H}\varphi = |f|^{q-2} f$ , that is,  $f$  solves (5.4). By Theorem 4.4 we can assume that  $f$  coincides with the function  $F = G' + HG$ . Hence  $g' + Hg = G' + HG$ , that is,  $g = G$  and the theorem is completely proved.  $\square$

## 6 Equivalent norms on $W^{k,p}(\mathbb{R})$

Here we improve Lemma 5.1 and Proposition 5.2 to include higher indexes  $k \geq 2$ . We start by introducing some notation.

Let  $m \geq 1$  be an integer and let

$$\vec{A} = (A_1, \dots, A_m) \in \mathbb{R}^m, \quad \vec{\gamma} = (\gamma_1, \dots, \gamma_m) \in \mathbb{R}^m$$

be given  $m$ -vectors. We define the  $m+1$  differential operators

$$L_h g = g'' - 2A_h g' - \gamma_h g, \quad \mathbb{L}_{\vec{A}, \vec{\gamma}} = L_1 \circ \dots \circ L_m,$$

so that  $\mathbb{L}_{\vec{A}, \vec{\gamma}}$  has order  $2m$ .

We distinguish the "even case"  $k = 2m$  from the "odd" one, when  $k = 2m + 1$ . For the proofs use induction. We omit details.

**Proposition 6.1** *Assume that  $A_h^2 + \gamma_h \geq 0$  for any  $h = 1, \dots, m$ . Then*

$$I_p(\vec{A}, \vec{\gamma}) := \inf_{\substack{g \in W^{2m,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |\mathbb{L}_{\vec{A}, \vec{\gamma}} g|^p ds}{\int_{\mathbb{R}} |g|^p ds} = \prod_{h=1}^m |\gamma_h|^p.$$

Moreover,  $\|g\|_{\mathbb{L}_{\vec{A}, \vec{\gamma}}} := \|\mathbb{L}_{\vec{A}, \vec{\gamma}} g\|_p$  is an equivalent norm on  $W^{2m}(\mathbb{R})$  provided that  $\gamma_h \neq 0$  for any  $h = 1, \dots, m$ .

**Proposition 6.2** *Assume that  $A_h^2 + \gamma_h \geq 0$  for any  $h = 1, \dots, m$  and let  $\lambda \in \mathbb{R}$ . Then*

$$M_p(\vec{A}, \vec{\gamma}; \lambda) := \inf_{\substack{g \in W^{2m+1,p}(\mathbb{R}) \\ g \neq 0}} \frac{\int_{\mathbb{R}} |\mathbb{L}_{\vec{A}, \vec{\gamma}} g' - \lambda \mathbb{L}_{\vec{A}, \vec{\gamma}} g|^p ds}{\int_{\mathbb{R}} |g|^p ds} = |\lambda|^p \prod_{h=1}^m |\gamma_h|^p.$$

Moreover,  $\|g\|_{\mathbb{L}_{\vec{A}, \vec{\gamma}}, B} := \|\mathbb{L}_{\vec{A}, \vec{\gamma}} g' - \lambda \mathbb{L}_{\vec{A}, \vec{\gamma}} g\|_p$  is an equivalent norm on  $W^{2m+1}(\mathbb{R})$  provided that  $\lambda \neq 0$  and  $\gamma_h \neq 0$  for any  $h = 1, \dots, m$ .

**Remark 6.3** *One can get more inequalities by taking advantage of the embeddings  $W^{k,p}(\mathbb{R}) \hookrightarrow W^{j,q}(\mathbb{R})$  for  $h = 1, \dots, k-1$  and  $q \geq p$ .*

## 7 The spaces $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$

In this section we assume that  $c_{\alpha,k,p}$  is positive and we study the properties of the Banach space  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ . We start with the lower order case  $k = 1$ .

### 7.1 The space $\mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha dx)$

If  $\alpha \neq p - n$ , then  $\mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha dx)$  is a well defined Banach space with norm

$$\|u\|_{1,\alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\nabla u|^p dx,$$

and it is continuously embedded into  $L^p(\mathbb{R}^n; |x|^{\alpha-2p} dx)$  by the Hardy inequality.

For  $g \in C_c^1(\mathbb{R})$  we put  $(\mathcal{T}_1 g)(x) = |x|^{-H_\alpha} g(-\log |x|)$ , where  $H_\alpha = \frac{n+\alpha}{p} - 1$ . Then clearly  $\mathcal{T}_1 : C_c^1(\mathbb{R}) \rightarrow C_{c,r}^1(\mathbb{R}^n \setminus \{0\})$  is a linear, invertible transform.

**Lemma 7.1** *Assume  $\alpha \neq p - n$ .*

- i) The transform  $\mathcal{T}_1$  can be extended in a unique way to a bicontinuous isomorphism  $W^{1,p}(\mathbb{R}) \rightarrow \mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha dx)$ .*
- ii) If  $\alpha > p - n$  then  $C_{c,r}^1(\mathbb{R}^n) \subset \mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha dx)$ . In particular, if  $p < n$  then  $\mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^0 dx) = \mathcal{D}_r^{1,p}(\mathbb{R}^n)$ .*

**Proof.** Notice that

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla(\mathcal{T}_1 g)|^p dx = \omega_n \int_{\mathbb{R}} |g' + H_\alpha g|^p ds$$

for any  $g \in C_c^1(\mathbb{R})$ . Therefore *i)* follows from Proposition 4.2, as  $H_\alpha \neq 0$ .

To prove *ii)*, take any  $u \in C_{c,r}^1(\mathbb{R}^n)$ , and let  $g = \mathcal{T}_1^{-1}u$ . Then  $g \equiv 0$  for  $s \ll 0$  and  $g(s), g'(s) = O(e^{-H_\alpha s})$  for  $s \rightarrow \infty$ . Since  $H_\alpha > 0$  then  $g$  and  $g'$  decay exponentially at infinity, and therefore  $g \in W^{1,p}(\mathbb{R})$ . Thus  $u \in \mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha)$  by *i)*, as desired.  $\square$

The next theorem was proved by Catrina and Wang in [9] when  $p = 2$ . Even if it could be already known also in the non-Hilbertian case  $p \neq 2$ , we provide here a proof that is based on Theorem 4.4. We identify functions that coincide up to a rescaling and a change of sign.

**Theorem 7.2** *If  $\alpha \neq p - n$  and  $q > p$ , then problem (1.12) has a unique nontrivial radial solution  $U$  in  $\mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha dx)$ . Moreover,  $U$  achieves the best constant  $S_{1,q,0}(\alpha)$  and it is given by*

$$U(|x|) = \left( \frac{q(p-1)}{p} \left| \frac{n-p+\alpha}{p-1} \right|^p \right)^{\frac{1}{q-p}} \left( 1 + |x|^{\frac{(n-p+\alpha)(q-p)}{p(p-1)}} \right)^{\frac{p}{p-q}}.$$

**Proof.** Using the definitions and the results in Section 4, it is easy to compute

$$S_{1,q,0}(\alpha) = \inf_{\substack{u \in \mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^\alpha dx) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\nabla u|^p dx}{\left( \int_{\mathbb{R}^n} |x|^{-\beta_{1,q}} |u|^q dx \right)^{p/q}} = \omega_n^{\frac{q-p}{q}} M_{p,q}(-H_\alpha),$$

where  $\beta_{1,q}$  is defined in (1.4). Moreover,  $u = \mathcal{T}_1 g$  solves (1.12) if and only if  $g$  is a weak solution to (4.4), where  $\lambda = -H_\alpha$ . The conclusion follows by Theorem 4.4.  $\square$

## 7.2 The space $\mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$

In order to simplify notation we put

$$H_2 = H_{\alpha,2} = \frac{n+\alpha}{p} - 2, \quad \gamma_2 = \gamma_{\alpha,2} = \left( \frac{n+\alpha}{p} - 2 \right) \left( n - \frac{n+\alpha}{p} \right),$$

compare with (3.2) and (1.6). We need also the constant

$$A_2 = \frac{n-2}{2} - H_2.$$

Notice that

$$A_2^2 + \gamma_2 = \left( \frac{n-2}{2} \right)^2 \geq 0, \quad H_2^2 + 2A_2 H_2 - \gamma_2 = 0. \quad (7.1)$$

Assume that  $\gamma_2 \neq 0$ , that is,  $\alpha \notin \{2p - n, np - n\}$ . Then  $\mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$  is a Banach space with norm

$$\|u\|_{2,\alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx.$$

Moreover,  $\mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$  is continuously embedded into  $L^p(\mathbb{R}^n; |x|^{\alpha-2p} dx)$  and into  $\mathcal{D}_r^{1,p}(\mathbb{R}^n; |x|^{\alpha-p} dx)$  by (1.8), (2.4).

For  $g \in C_c^2(\mathbb{R})$  we put  $(\mathcal{T}_2 g)(x) = |x|^{-H_2} g(-\log |x|)$ . Then  $\mathcal{T}_2$  is a linear, invertible transform  $C_c^2(\mathbb{R}) \rightarrow C_{c,r}^2(\mathbb{R}^n \setminus \{0\})$ .

Now we prove the second-order version of Lemma 7.1.

**Lemma 7.3** *Assume  $\alpha \notin \{2p - n, np - n\}$ .*

- i) *The transform  $\mathcal{T}_2$  can be extended in a unique way to a bicontinuous isomorphism  $W^{2,p}(\mathbb{R}) \rightarrow \mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$ .*
- ii) *if  $\alpha > 2p - n$  then  $C_{c,r}^2(\mathbb{R}^n) \subset \mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$ . In particular, if  $2p < n$  then  $\mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^0 dx) = \mathcal{D}_r^{2,p}(\mathbb{R}^n)$ .*

**Proof.** By direct computation one can check that

$$\int_{\mathbb{R}^n} |x|^\alpha |\Delta(\mathcal{T}_2 g)|^p dx = \omega_n \int_{\mathbb{R}} |g'' - 2A_2 g' - \gamma_2 g|^p ds$$

for any  $g \in C_{c,r}^1(\mathbb{R})$ . Therefore Proposition 5.2 and Lemma 7.1 immediately imply i). To prove ii) fix  $u \in C_{c,r}^2(\mathbb{R}^n)$ , put  $g = \mathcal{T}_2^{-1}u$  and then argue as in Lemma 7.3.  $\square$

By density we have that

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{-\beta_{2,q}} |u|^q dx &= \omega_n \int_{\mathbb{R}} |g|^q ds \\ \int_{\mathbb{R}^n} |x|^{-\beta_{1,q}} |\nabla u|^q dx &= \omega_n \int_{\mathbb{R}} |g' + H_2 g|^q ds \\ \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx &= \omega_n \int_{\mathbb{R}} |g'' - 2A_2 g' - \gamma_2 g|^p ds \end{aligned}$$

for any  $u = \mathcal{T}_2 g \in \mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$ ,  $q \geq p$ , where, accordingly with (1.4),

$$\beta_{2,q} = n - q \frac{n - 2p + \alpha}{p}, \quad \beta_{1,q} = n - q \frac{n - p + \alpha}{p}.$$

Now we fix an exponent  $q > p$  and we use Lemma 7.3 together with the results in Section 5 to study the best constants (1.13) and (1.14), that will be discuss separately.

**Theorem 7.4** *If  $\alpha \notin \{2p - n, np - n\}$  and  $q > p$ , then  $S_{2,q,0}(\alpha)$  is positive and attained in  $\mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$ .*

**Proof.** Notice that  $S_{2,q,0}(\alpha) = \omega_n^{\frac{q-p}{p}} I_{p,q}(A_2, \gamma_2)$  and apply Proposition 5.2.  $\square$

We conclude this section with the complete classification of extremals of  $S_{2,q,1}(\alpha)$ . As before, we identify functions that coincide up to a rescaling and a change of sign.

**Theorem 7.5** *If  $\alpha \notin \{2p-n, np-n\}$  and  $q > p$ , then  $S_{2,q,1}(\alpha)$  is positive. Moreover, equation*

$$\Delta(|x|^\alpha |\Delta U|^{p-2} \Delta U) + \operatorname{div}(|x|^{-\beta_{1,q}} |\nabla U|^{q-2} \nabla U) = 0$$

*has a unique nontrivial solution  $U \in \mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$ . More precisely,  $U$  achieves the best constant  $S_{2,q,1}(\alpha)$  and it is given by*

$$U(s) = k \int_{|x|}^{\infty} t^{1-\frac{\alpha}{p-1}} \left(1 + t^{\frac{(np-n-\alpha)(q-p)}{p(p-1)}}\right)^{\frac{p}{p-q}} dt \quad \text{if } \alpha > 2p-n$$

$$U(s) = k \int_0^{|x|} t^{1-\frac{\alpha}{p-1}} \left(1 + t^{\frac{(np-n-\alpha)(q-p)}{p(p-1)}}\right)^{\frac{p}{p-q}} dt \quad \text{if } \alpha < 2p-n,$$

where

$$k = \left(\frac{q}{p} \frac{1}{(p-1)^{p-1}}\right)^{\frac{1}{q-p}} |np-n-\alpha|^{\frac{p}{q-p}}.$$

**Proof.** Use the transform  $\mathcal{T}_2$  to check that  $S_{2,q,1}(\alpha) = \omega_n^{\frac{q-p}{q}} J_{p,q}(A_2, \gamma_2, H_2)$ . Thus  $S_{2,q,1}(\alpha) > 0$  by *iii*) in Proposition 5.4. The conclusion of the proof is immediate, from Theorem 5.7.  $\square$

**Remark 7.6** *Here we deal with the infima*

$$S_{2,q,0}(\alpha) = \inf_{\substack{u \in C_{c,r}^2(\mathbb{R}^n \setminus \{0\}) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx}{\left(\int_{\mathbb{R}^n} |x|^{-n+q\frac{n-2p+\alpha}{p}} |u|^q dx\right)^{p/q}}$$

$$S_{2,q,1}(\alpha) := \inf_{\substack{u \in C_{c,r}^2(\mathbb{R}^n \setminus \{0\}) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx}{\left(\int_{\mathbb{R}^n} |x|^{-n+q\frac{n-p+\alpha}{p}} |\nabla u|^q dx\right)^{p/q}}$$

in the "degenerate" cases  $\alpha \in \{2p-n, np-n\}$ . Thanks to the Emden-Fowler transform  $\mathcal{T}_2 : C_c^2(\mathbb{R}) \rightarrow C_{c,r}^2(\mathbb{R}^n \setminus \{0\})$ , it is easy to check that

$$S_{2,q,0}(\alpha) = \omega_n^{\frac{q-p}{p}} I_{p,q}(A_2, \gamma_2), \quad S_{2,q,1}(\alpha) = \omega_n^{\frac{q-p}{p}} J_{p,q}(A_2, \gamma_2, H_2).$$



Using Lemma 5.3 and Proposition 5.4 one can prove the following facts:

- i)  $S_{2,q,0}(np - n) = S_{2,q,0}(2p - n) = 0$ ,
- ii)  $S_{2,q,1}(np - n) = 0$ ,
- iii)  $S_{2,q,1}(2p - n) = M_{p,q}(n - 2)$  and no function in  $L^p(\mathbb{R}^n; |x|^{\alpha-2p} dx)$  achieves  $S_{2,q,1}(2p - n)$ . Note that  $S_{2,q,1}(2p - n)$  vanishes if  $n = 2$ , it is positive and explicitly known if  $n \geq 3$ , see Theorem 4.4.

**Remark 7.7** Assume  $\alpha > 2p - n$ ,  $\alpha \neq n(p - 1)$ . Then the weight  $|x|^{\alpha-2p}$  is locally integrable and the space  $C_{c,r}^2(\mathbb{R}^n)$  is dense in  $\mathcal{D}_r^{2,p}(\mathbb{R}^n; |x|^\alpha dx)$  by ii) in Lemma 7.3. In particular, inequalities (1.8) and (2.4) hold in  $C_{c,r}^2(\mathbb{R}^n)$ . We also have that

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx &\geq S_{2,q,0}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-n+q\frac{n-2p+\alpha}{p}} |u|^q dx \right)^{p/q} \quad \forall u \in C_{c,r}^2(\mathbb{R}^n) \\ \int_{\mathbb{R}^n} |x|^\alpha |\Delta u|^p dx &\geq S_{2,q,1}(\alpha) \left( \int_{\mathbb{R}^n} |x|^{-n+q\frac{n-p+\alpha}{p}} |\nabla u|^q dx \right)^{p/q} \quad \forall u \in C_{c,r}^2(\mathbb{R}^n). \end{aligned}$$

**Remark 7.8** Assume  $n > p$ ,  $n \neq 2p$ , and take  $\alpha = 0$ ,  $q = p^* = \frac{np}{n-p}$ . From Theorem 7.5 we infer that infimum  $S_{2,p}^*$  in (1.16) is achieved by the function

$$\begin{aligned} U(x) &= \int_{|x|}^{\infty} s \left( 1 + s^{p^*} \right)^{\frac{p-n}{p}} ds \quad \text{if } n > 2p, \\ U(x) &= \int_0^{|x|} s \left( 1 + s^{p^*} \right)^{\frac{p-n}{p}} ds \quad \text{if } n < 2p. \end{aligned}$$

If  $n = 2p$  then  $p^* = n$ , the constant  $S_{2,n,1}(0)$  in Remark 7.6 is positive and any primitive  $U^*$  of the function  $ks(1 + s^n)^{-1}$  is indeed a solution to the Euler-Lagrange equation (1.15). Moreover,  $U^*$  satisfies

$$\int_{\mathbb{R}^n} |\Delta U^*|^p dx < \infty, \quad \int_{\mathbb{R}^n} |x|^{-p} |\nabla U^*|^p dx < \infty, \quad \int_{\mathbb{R}^n} |x|^{-2p} |U^*|^p dx = \infty.$$

For instance, if  $p = 2$  and  $n = 4$  then  $U^*(x) = 4\sqrt{2} \arctan r^2 + \text{const.}$

### 7.3 Higher order spaces and inequalities

To simplify notation we put

$$H_h = H_{\alpha,h} = \frac{n + \alpha}{p} - h, \quad \gamma_h = \gamma_{\alpha,h} = H_h(n - 2 - H_h)$$

for any integer  $h \geq 1$ , compare with (3.2) and (1.6). We introduce also the constants

$$A_h = \frac{n-2}{2} - H_h.$$

Notice that

$$A_h^2 + \gamma_h = \left( \frac{n-2}{2} \right)^2 \geq 0, \quad H_h^2 + 2A_h H_h - \gamma_h = 0.$$

In this section we will always assume that (3.1) is satisfied. Thus  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  is a well defined Banach space with norm

$$\|u\|_{k,\alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx,$$

and it is continuously embedded into  $L^p(\mathbb{R}^n; |x|^{\alpha-kp} dx)$  by Theorem 1.1. In addition, for any integer  $j \in \{1, \dots, k-1\}$  it turns out that  $c_{\alpha-(k-j)p,j,p} > 0$ . Thus the space  $D_r^{j,p}(\mathbb{R}^n; |x|^{\alpha-(k-j)p} dx)$  is well defined. Finally,  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  is continuously embedded into  $D_r^{j,p}(\mathbb{R}^n; |x|^{\alpha-(k-j)p} dx)$  by Remark 2.3.

Accordingly with Section 3, we put  $(\mathcal{T}_k g)(x) = |x|^{-H_k} g(-\log|x|)$  for  $g \in C_c^k(\mathbb{R})$ .

**Lemma 7.9** *The transform  $\mathcal{T}_k$  can be extended in a unique way to a bicontinuous isomorphism  $W^{k,p}(\mathbb{R}) \rightarrow \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ .*

**Proof.** Put  $(\mathcal{T}_0 g)(x) = |x|^{-\frac{n+\alpha}{p}}$  for  $g \in C_c^k(\mathbb{R})$ . Notice that

$$\Delta(\mathcal{T}_h g) = \mathcal{T}_{h-2} (g'' - 2A_h g' - \gamma_h g) \quad (7.2)$$

for any  $h \geq 2$  (use an induction argument).

We distinguish the case  $k = 2m$  from the case of odd order operators.

**Even poliharmonic operators.** Assume that  $k = 2m$  is an even integer and that  $\gamma_{2h} \neq 0$  for any  $h = 1, \dots, m$ . We adopt the notation in Section 6 with

$$\begin{aligned} \vec{A} &= (A_2, \dots, A_{2m}), \quad \vec{\gamma} = (\gamma_2, \dots, \gamma_{2m}), \\ L_h g &= g'' - 2A_{2h} g' - \gamma_{2h} g, \quad \mathbb{L}_{\vec{A}, \vec{\gamma}} = L_1 \circ \dots \circ L_m. \end{aligned}$$

Using (7.2) it is easy to prove by induction that

$$\Delta^m(\mathcal{T}_{2m} g) = \mathcal{T}_0(\mathbb{L}_{\vec{A}, \vec{\gamma}} g)$$

for any  $g \in C_c^k(\mathbb{R})$ . Therefore, for  $u = \mathcal{T}_{2m}(g)$  the following equality holds

$$\|u\|_{2m,p,\alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\Delta^m u|^p dx = \omega_n \int_{\mathbb{R}} \left| \mathbb{L}_{\vec{A}, \vec{\gamma}} g \right|^p ds.$$

The conclusion in the even case follows by Proposition 6.1.

**Odd poliharmonic operators.** When  $k = 2m+1$  is odd we assume that  $\gamma_{2h+1} \neq 0$  for any  $h = 1, \dots, m$  and that the Hardy constant is positive. Now we put

$$\begin{aligned} \vec{A} &= (A_3, \dots, A_{2m+1}), \quad \vec{\gamma} = (\gamma_3, \dots, \gamma_{2m+1}), \\ L_h g &= g'' - 2A_{2h+1} g' - \gamma_{2h+1} g, \quad \mathbb{L}_{\vec{A}, \vec{\gamma}} = L_1 \circ \dots \circ L_m. \end{aligned}$$

Using (7.2) one can prove by induction that

$$\Delta^m(\mathcal{T}_{2m+1}g) = \mathcal{T}_1(\mathbb{L}_{\vec{A}, \vec{\gamma}} g)$$

for any  $g \in C_c^k(\mathbb{R})$ . Therefore, for  $u = \mathcal{T}_{2m+1}g$  it is easy to compute

$$\|u\|_{2m+1,p,\alpha}^p = \int_{\mathbb{R}^n} |x|^\alpha |\nabla(\Delta^m u)|^p dx = \omega_n \int_{\mathbb{R}} \left| \mathbb{L}_{\vec{A}, \vec{\gamma}} g' + H_\alpha \mathbb{L}_{\vec{A}, \vec{\gamma}} g \right|^p ds.$$

The conclusion follows by Proposition 6.2.  $\square$

**Remark 7.10** *The computations in the proof of Lemma 7.9 together with Propositions 6.1, 6.2 provide an alternative proof of Theorem 1.1.*

**Remark 7.11** *Assume  $\alpha > kp - n$ . One can adapt the arguments already used in the lowest order case  $k = 1$  to show that  $C_{c,r}^k(\mathbb{R}^n) \subset \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ .*

**Remark 7.12** *Thanks to Lemma 7.9 we can identify  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  with the Sobolev space  $W^{k,p}(\mathbb{R})$  through  $\mathcal{T}_k$ . In particular, every function  $u \in \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  has continuous derivatives up to the order  $k-1$ , and partial derivatives of order  $k$  (in the classical sense) exist for almost every  $|x| > 0$ .*

Now we focus our attention on Sobolev type embeddings. For  $j \in \{0, \dots, k\}$  and  $q > p$  define  $\beta_{k-j,q}$  accordingly with (1.4).

**Remark 7.13** Fix an index  $j = 1, \dots, k-1$ , and notice that

$$\tilde{H}_h := \frac{n + \beta_{k-j,q}}{q} - h = \frac{n + \alpha}{p} - (k - j + h) = H_{k-j+h},$$

so that  $\tilde{H}_j = H_k$ . Moreover,  $\tilde{\gamma}_h := \tilde{H}_h(n - 2 + \tilde{H}_h)$  satisfy (3.1) with  $\alpha, k, p$  replaced by  $\beta_{k-j,q}, j$  and  $q$ , respectively. Thus  $\mathcal{D}_r^{j,q}(\mathbb{R}^n; |x|^{-\beta_{k-j,p}} dx)$  is a well defined Banach space, with norm

$$\left( \int_{\mathbb{R}^n} |x|^{-\beta_{k-j,q}} |\nabla^j u|^q dx \right)^{1/q},$$

and  $\mathcal{T}_k$  can be used to identify  $W^{j,q}(\mathbb{R})$  with  $\mathcal{D}_r^{j,q}(\mathbb{R}^n; |x|^{-\beta_{k-j,p}} dx)$ .

We are in position to state the main result of this section.

**Theorem 7.14** If  $q > p > 1$  and  $j \in \{0, 1, \dots, k-1\}$ , then the infimum  $S_{k,q,j}(\alpha)$  in (1.11) is positive and achieved.

**Proof.** By Lemma 7.9, Sobolev embedding theorem and Remark 7.13, we have the following chain of continuous arrows:

$$\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \xrightarrow{\mathcal{T}_k^{-1}} W^{k,p}(\mathbb{R}) \hookrightarrow W^{j,q}(\mathbb{R}) \xrightarrow{\mathcal{T}_k} \mathcal{D}_r^{j,q}(\mathbb{R}^n; |x|^{-\beta_{k-j,q}} dx).$$

In particular,  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  is continuously embedded into  $\mathcal{D}_r^{j,q}(\mathbb{R}^n; |x|^{-\beta_{k-j,q}} dx)$  and hence  $S_{k,q,j}(\alpha)$  is positive.

To prove that  $S_{k,q,j}(\alpha)$  is achieved one can study an equivalent minimization problem for functions in  $W^{k,p}(\mathbb{R})$ , as we did in Section 7.2 for the case  $k = 2$ . Here we adopt a direct approach. The strategy is essentially the same of [17], and it was originally inspired to the author by the famous paper [21] by Sacks and Uhlenbeck.

We introduce the energies  $e_{k,p}, e_{j,q}$  and  $E : \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx) \rightarrow \mathbb{R}$  by

$$e_{k,p}(u) = \frac{1}{p} \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx, \quad e_{j,q}(u) = \frac{1}{q} \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u|^q dx,$$

$$E(u) = e_{k,p}(u) - e_{j,q}(u),$$

where  $\beta = \beta_{k-j,q}$ . The functional  $E$  is of class  $C^1$  and minimizers for  $\tilde{S} := S_{k,q,j}(\alpha)$  give rise to critical points of  $E$ . Using Ekeland's variational principle we can select a minimizing sequence  $u_h$  such that

$$E'(u_h) \cdot v \rightarrow 0 \quad \text{uniformly for } v \text{ in bounded sets of } \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx). \quad (7.3)$$

In particular we have  $o(\|u_h\|_{k,\alpha}) = E'(u_h) \cdot u_h = \|u_h\|_{k,\alpha}^p + O(\|u_h\|_{k,\alpha}^q)$ . Thus  $u_h$  is bounded and

$$\int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^q dx = \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx + o(1) = \tilde{S}^{\frac{q}{q-p}} + o(1).$$

Since the ratio in (1.11) is invariant with respect to dilations, we can assume that

$$\int_{\{|x|<2\}} |x|^{-\beta} |\nabla^j u_h|^q dx = \left(\frac{1}{2} \tilde{S}\right)^{\frac{q}{q-p}}. \quad (7.4)$$

We have to prove that, up to a subsequence,  $u_h$  converges weakly to some nontrivial limit  $u$ . Then, a standard convexity argument shows that  $u$  achieves  $\tilde{S}$ . Assume by contradiction that  $u_h \rightharpoonup 0$  weakly in  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ . By Lemma 8.2 we have that  $|\nabla^j u_h| \rightarrow 0$  in  $L_{\text{loc}}^q(\mathbb{R}^n \setminus \{0\})$  and therefore

$$\int_{\{|x|<1\}} |x|^{-\beta} |\nabla^j u_h|^q dx = \left(\frac{1}{2} \tilde{S}\right)^{\frac{q}{q-p}} + o(1) \quad (7.5)$$

by (7.4). In essence, the idea is to fix a function  $\varphi \in C_{c,r}^k(\mathbb{R}^n)$  such that  $0 \leq \varphi \leq 1$ ,  $\varphi \equiv 1$  on the unit ball, and to use  $E'(u_h) \cdot (\varphi^p u_h) = o(1)$  to reach a contradiction with (7.5). However this can not be done if  $p < k$ , as  $\varphi^p$  is not of class  $C^k(\mathbb{R}^n)$ . Thus we define

$$\Phi_\varepsilon(x) = (\varepsilon^2 + \varphi(x)^2)^{p/2} - \varepsilon^p,$$

where  $\varepsilon \in (0, 1)$  is fixed. Notice that  $\Phi_\varepsilon \in C_{c,r}^k(\mathbb{R}^n)$  and that  $\Phi_\varepsilon$  is a constant in a neighborhood of 0. Thanks to Lemma 8.1 we can compute  $E'(u_h) \cdot (\Phi_\varepsilon u_h)$ . Since the family  $\Phi_\varepsilon u_h$  is uniformly bounded in  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  as  $h \rightarrow \infty$ , then (7.3) gives

$$e'_{k,p}(u) \cdot (\Phi_\varepsilon u_h) = e'_{j,q}(u) \cdot (\Phi_\varepsilon u_h) + o(1). \quad (7.6)$$

We apply (8.2) in Lemma 8.4 to get

$$\begin{aligned} e'_{k,p}(u) \cdot (\Phi_\varepsilon u_h) &= \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^{p-2} \nabla^k u_h \cdot \nabla^k (\Phi_\varepsilon u_h) dx \\ &= \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^p \Phi_\varepsilon dx + o(1). \end{aligned}$$

From  $\Phi_\varepsilon \geq \varphi^p - \varepsilon^p$ , we infer

$$\begin{aligned} e'_{k,p}(u) \cdot (\Phi_\varepsilon u_h) &\geq \int_{\mathbb{R}^n} |x|^\alpha |\varphi \nabla^k u_h|^p dx - c\varepsilon^p + o(1) \\ &= \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k (\varphi u_h)|^p dx - c\varepsilon^p + o(1) \end{aligned}$$

by (8.1), where  $c = \sup_h \|u_h\|_{k,\alpha}$ . Since  $\varphi u_h \in \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  by Lemma 8.1), from the definition of  $\tilde{S}$  we get

$$e'_{k,p}(u) \cdot (\Phi_\varepsilon u_h) \geq \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j(\varphi u_h)|^q dx \right)^{p/q} - c\varepsilon^p + o(1). \quad (7.7)$$

To estimate the right hand side of (7.6) we use (8.1) with  $\alpha, k, p$  replaced by  $\beta, j, q$ , respectively, to get

$$\begin{aligned} e'_{j,q}(u) \cdot (\Phi_\varepsilon u_h) &= \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^{q-2} \nabla^j u_h \cdot \nabla^j (\Phi_\varepsilon u_h) dx \\ &= \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^{q-p} |\nabla^j u_h|^p \Phi_\varepsilon dx + o(1) \\ &\leq \left( \int_{\{|x|<2\}} |x|^{-\beta} |\nabla^j u_h|^q dx \right)^{\frac{q-p}{p}} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j u_h|^q (\Phi_\varepsilon)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} + o(1) \end{aligned}$$

by Hölder inequality. From (7.4), Lemma 8.5 and (8.1) we obtain

$$e'_{j,q}(u) \cdot (\Phi_\varepsilon u_h) \leq \frac{1}{2} \tilde{S} \left( \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j(\varphi u_h)|^q dx \right)^{p/q} + c\varepsilon^{p/q} + o(1). \quad (7.8)$$

Comparing (7.6), (7.7) and (7.8) we conclude that

$$\int_{\{|x|<1\}} |x|^{-\beta} |\nabla^j u_h|^q dx \leq \int_{\mathbb{R}^n} |x|^{-\beta} |\nabla^j(\varphi u_h)|^q dx \leq c\varepsilon + o(1),$$

since  $\varphi \equiv 1$  on the unit ball. Letting  $h \rightarrow \infty$  we readily get a contradiction with (7.5), as  $\varepsilon > 0$  was arbitrarily chosen. The proof is complete.  $\square$

**Remark 7.15** *Theorem 7.14 includes  $k$  Sobolev constants without weights. Assume  $n > kp$ , take an integer  $j \in \{0, \dots, k-1\}$  and  $\alpha = 0$ . In addition, assume that  $q$  equals the  $(k-j)$ th order critical exponent*

$$p^{(k-j)*} := \frac{np}{n - (k-j)p}.$$

*Then  $\beta_{k-j,q} = 0$ . Taking Remark 7.11 into account, by Theorem 7.14 we have that the radial Sobolev constant*

$$S_{k,p}^{(k-j)*} := \inf_{\substack{u \in \mathcal{D}_r^{k,p}(\mathbb{R}^n) \\ u \neq 0}} \frac{\int_{\mathbb{R}^n} |\nabla^k u|^p dx}{\left( \int_{\mathbb{R}^n} |\nabla^j u|^{\frac{pn}{n-(k-j)p}} dx \right)^{\frac{n-(k-j)p}{n}}}$$

*is positive and achieved in  $\mathcal{D}_r^{k,p}(\mathbb{R}^n)$ .*

## 8 Appendix

Here we prove some compactness and technical results that have been used in the proof of Theorem 7.14. We always assume (3.1) is satisfied.

**Lemma 8.1** *Let  $u \in \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  and  $\Phi \in C_{c,r}^k(\mathbb{R}^n)$  such that  $\Phi$  is constant in a neighborhood of 0. Then  $\Phi u \in \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  and*

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k(\Phi u)|^p dx \leq c_\Phi \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx,$$

where the constant  $c_\Phi$  does not depend on  $u$ .

**Proof.** Let  $g = \mathcal{T}_k^{-1}u \in W^{k,p}(\mathbb{R})$  and put  $\tilde{\Phi}(s) := \Phi(e^{-s}) \in C^k(\mathbb{R})$ . Notice that  $\tilde{\Phi}(s) \equiv 0$  for  $s \ll 0$ ,  $\tilde{\Phi}(s)$  is a constant for  $s \gg 0$ . Thus  $\tilde{\Phi}g \in W^{k,p}(\mathbb{R})$  and hence  $\Phi u = \mathcal{T}_k(\tilde{\Phi}g) \in \mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ . Finally, from Lemma 7.9 we infer that

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k(\Phi u)|^p dx \leq c \|\tilde{\Phi}g\|_{W^{k,p}}^p \leq c_\Phi \|g\|_{W^{k,p}}^p \leq c_\Phi \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u|^p dx,$$

and the Lemma is proved.  $\square$

Next we need few results on weak convergence in  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$ . Notice that  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  is reflexive because it is topologically equivalent to  $W^{k,p}(\mathbb{R})$  (or because its norm is uniformly convex).

**Lemma 8.2** *Let  $\Omega$  be a domain such that  $\overline{\Omega} \subset \mathbb{R}^n \setminus \{0\}$ . Then  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  is compactly embedded into  $W^{k-1,\tau}(\Omega)$  for any  $\tau \geq p$ .*

**Proof.** Use the Emden-Fowler transform and Rellich theorem for  $W^{k,p}(I)$ , where  $I$  is an appropriate bounded interval.  $\square$

**Remark 8.3** *Lemma 8.2 is closely related to Theorem II.1 in [14]. Actually, the same argument shows that  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  is compactly embedded into  $C^{k-1}(\overline{\Omega})$ .*

**Lemma 8.4** *Let  $\Phi \in C_{c,r}^k(\mathbb{R}^n)$  be a given function, such that  $\Phi$  is constant in a neighborhood of 0. If  $u_h \rightharpoonup 0$  in  $\mathcal{D}_r^{k,p}(\mathbb{R}^n; |x|^\alpha dx)$  then*

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k(\Phi u) - \Phi \nabla^k u|^p dx = o(1) \tag{8.1}$$

$$\int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^{p-2} \nabla^k u_h \cdot \nabla^k(\Phi u_h) dx = \int_{\mathbb{R}^n} |x|^\alpha |\nabla^k u_h|^p \Phi dx + o(1). \tag{8.2}$$

**Proof.** Note that  $\nabla^k(\Phi u) = \Phi \nabla^k u + \Psi D_{k-1} u$ , where  $\Psi \in C_c^0(\mathbb{R}^n \setminus \{0\})$  and  $D_{k-1}$  is a differential operator of order  $k-1$ . Thus (8.1) holds by Lemma 8.2. To prove (8.2) use (8.1) and Hölder inequality.  $\square$

We conclude the preliminaries with an elementary lemma.

**Lemma 8.5** *Let  $\varepsilon, \varphi \in [0, 1]$ ,  $1 < p < q$ . Then there exists a constant  $c \geq 0$  such that  $\left[(\varepsilon^2 + \varphi^2)^{p/2} - \varepsilon^p\right]^{q/p} \leq \varphi^q + c\varepsilon$ .*

**Proof.** Fix  $\varphi$  and put  $\psi(\varepsilon) = \left[(\varepsilon^2 + \varphi^2)^{p/2} - \varepsilon^p\right]^{q/p}$ . If  $p \leq 2$  then  $\psi$  is non increasing and hence the conclusion in the lemma holds with  $c = 0$ . If  $2 < p < q$  it suffices to notice that  $\psi$  is differentiable at  $\varepsilon = 0$ .  $\square$

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